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Charged bosons and the coherent state

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Abstract. The coherent state for charged bosons is constructed, its properties are investigated and the corresponding classical model is discussed.

1. Introduction

The coherent state was first constructed (Schrödinger 1926, Glauber 1963) for the simple harmonic oscillator. The Hamiltonian of the system

$$H = p^2/2m + \tfrac{1}{2}m\omega^2x^2 \tag{1}$$

may be rewritten as

$$H = \hbar\omega(a^\dagger a + \tfrac{1}{2}) \tag{2}$$

by defining annihilation and creation operators

$$a = (p - \mathrm{i}m\omega x)/(2m\omega\hbar)^{1/2} \qquad a^\dagger = (p + \mathrm{i}m\omega x)/(2m\omega\hbar)^{1/2}. \tag{3}$$

The eigenstates of the Hamiltonian, $|n\rangle$, belonging to the energy eigenvalue $E_n = \hbar\omega(n + \frac{1}{2})$, where n is a non-negative integer, may then be obtained with the properties

$$a^\dagger a|n\rangle = n|n\rangle \qquad a^\dagger|n\rangle = (n+1)^{1/2}|n+1\rangle \qquad a|n\rangle = n^{1/2}|n-1\rangle. \tag{4}$$

The coherent state may then be constructed out of these states thus

$$|\alpha\rangle = \exp(-\tfrac{1}{2}|\alpha|^2)\sum_{n=0}^{\infty}\frac{\alpha^n}{(n!)^{1/2}}|n\rangle \tag{5}$$

where α is a complex number and the factor outside the summation is the normalization constant. The coherent state $|\alpha\rangle$ is an eigenstate of the annihilation operator a, namely

$$a|\alpha\rangle = \alpha|\alpha\rangle. \tag{6}$$

The coherent state may also be written in the form

$$|\alpha\rangle = \exp(-\alpha^* a + \alpha a^\dagger)|0\rangle \tag{7}$$

and is thus 'a displacement of the vacuum'. The coherent states form a complete (albeit an overcomplete) set in the sense that

$$\int \frac{\mathrm{d}^2\alpha}{\pi}|\alpha\rangle\langle\alpha| = 1 \tag{8}$$

where the integration is over the whole complex α plane. The coherent state constitutes a state of minimum uncertainty, namely

$$\Delta p \Delta X = \tfrac{1}{2}\hbar. \tag{9}$$

Since the coherent state is a non-stationary state it develops with time in an interesting manner and, taking $\alpha(t=0) = \lambda\, e^{-i\theta}$, it follows that

$$\langle \alpha, t|X|\alpha, t\rangle = \left[2\lambda\left(\frac{\hbar}{2m\omega}\right)^{1/2}\right]\sin(\omega t+\theta). \tag{10}$$

Identifying the constant in square brackets with the amplitude, the expectation value of the displacement in the coherent state behaves like the displacement of a classical oscillator. In this sense the coherent state is called a 'classical state'.

The coherent state has found widespread applications (Glauber 1969, Klauder and Sudarshan 1968, Perina 1971) in nonlinear optics and laser physics, in the discussion of the superfluid state (Langer 1969) and in nuclear physics (Bhaumik *et al* 1975). However, in all these cases the quanta involved are uncharged and the type of coherent state discussed above is adequate. Recent attempts (Botke *et al* 1974) to use the coherent state basis for the description of pion production in high-energy collisions necessitate the present discussion of the coherent state for charged bosons. We therefore discuss in the present paper the construction, properties and corresponding classical model of the coherent state for bosons possessing some 'charge' which is absolutely conserved.

2. Coherent state for charged bosons

The coherent state is a superposition of states containing different numbers of quanta phase-locked in the manner depicted in equation (5). However, if these quanta possess some absolutely conserved 'charge' Q it is impossible to construct coherent superpositions of states with different values of Q or to measure the corresponding phases. This is the content of the superselection rule (Wick *et al* 1952). Thus coherent states for charged quanta need careful consideration.

Let us introduce 'charge' by defining two types of quanta possessing 'charge' $+1$ and -1 with corresponding annihilation operators a and b. Thus

$$[a, a^{\dagger}] = 1 = [b, b^{\dagger}] \qquad [a, a] = 0 = [b, b] \qquad [a, b] = 0 = [a^{\dagger}, b] \tag{11}$$

and the charge operator is given by

$$Q = a^{\dagger}a - b^{\dagger}b. \tag{12}$$

Clearly the charge operator Q does not commute with a or with b. Therefore we cannot demand that the coherent state be simultaneously an eigenstate of the charge and the annihilation operators a or b. Nevertheless, in view of the fact that

$$[Q, ab] = 0 \tag{13}$$

we may define the modified coherent state $|\xi, q\rangle$ for charged quanta to be simultaneously an eigenstate of Q and ab belonging to the eigenvalues q and ξ (a complex number) respectively; thus

$$Q|\xi, q\rangle = q|\xi, q\rangle \qquad ab|\xi, q\rangle = \xi|\xi, q\rangle. \tag{14}$$

This state can easily be constructed out of eigenstates $|n, m\rangle$ where

$$a^\dagger a|n, m\rangle = n|n, m\rangle \qquad b^\dagger b|n, m\rangle = m|n, m\rangle \tag{15}$$

to yield

$$|\xi, q\rangle = N_q \sum_{n=0}^{\infty} \frac{\xi^n}{[n!(n+q)!]^{1/2}} |n+q, n\rangle \tag{16}$$

where the normalization constant N_q is given by

$$N_q = \left(\sum_n \frac{(|\xi|^2)^n}{n!(n+q)!} \right)^{-1/2} = [(-\mathrm{i}|\xi|)^q J_q(2\mathrm{i}|\xi|)]^{-1/2} \tag{17}$$

where J_q is the Bessel function of order q. This coherent state may also be generated from the vacuum state (Schwinger 1965). Thus

$$F_q(a^\dagger b^\dagger \xi) a^{\dagger q}|0\rangle \tag{18a}$$

is the coherent state except for a normalization factor. The function F_q is given by

$$F_q(z) = q!(-z)^{-q/2} J_q(2\mathrm{i}z^{1/2}) = \sum_{n=0}^{\infty} \frac{q!}{n!(n+q)!} z^q. \tag{18b}$$

The above expressions are for $q > 0$ and analogous expressions for $q < 0$ are obtained by replacing a by b. These states constitute a complete set of states in the sense that

$$\sum_{q=0}^{\infty} \int \frac{\mathrm{d}^2\xi}{\pi} \phi_q(\xi) |\xi, q\rangle\langle \xi, q| = \mathbb{1} \tag{19}$$

where

$$\phi_q(\xi) = 2(-\mathrm{i})^q J_q(2\mathrm{i}|\xi|) K_q(2|\xi|) \tag{20a}$$

with

$$K_q(z) = \tfrac{1}{2}\pi\mathrm{i} \exp(\tfrac{1}{2}\mathrm{i}\pi q)(J_q(\mathrm{i}z) + \mathrm{i}N_q(\mathrm{i}z)). \tag{20b}$$

The coherence of the conventional coherent states (discussed in § 1) is realized through the correlation of signals from photon counters. For the coherent states defined here for charged quanta, the coherence is manifested by replacing the photon counters by detectors which respond to the simultaneous detection of positive and negative quanta.

The coherent state for charged quanta may also be obtained by projecting out a state of definite charge from the two-mode coherent state

$$|\alpha\beta\rangle = \exp(\alpha a^\dagger + \beta b^\dagger)|0\rangle = \sum_{n,m=0}^{\infty} \frac{\alpha^n \beta^m}{n!m!} a^{\dagger n} b^{\dagger m}|0\rangle \tag{21}$$

which does not have a definite charge. This is accomplished by putting

$$\alpha = \lambda\, \mathrm{e}^{-\mathrm{i}(\theta+\varphi)}, \qquad \beta = \mu\, \mathrm{e}^{-\mathrm{i}(\theta-\varphi)} \tag{22}$$

and observing that the state

$$|\lambda\mu\theta; q\rangle = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\varphi\, \mathrm{e}^{\mathrm{i}q\varphi} |\alpha\beta\rangle \tag{23}$$

is identical to the charged coherent state, equation (16), if we make the identification $\lambda\mu\, \mathrm{e}^{-2\mathrm{i}\theta} = \xi$.

Introducing the 'coordinates' corresponding to the a and b quanta

$$x_1=-\mathrm{i}\left(\frac{\hbar}{2m\omega}\right)^{1/2}(a^{\dagger}-a) \qquad x_2=-\mathrm{i}\left(\frac{\hbar}{2m\omega}\right)^{1/2}(b^{\dagger}-b) \tag{24}$$

it is easily seen that the average displacements of the two oscillators vanish and that

$$\langle\xi;q|x_1^2|\xi;q\rangle=\frac{\hbar}{2m\omega}\left(-2\mathrm{i}|\xi|\frac{J_{q+1}(2\mathrm{i}|\xi|)}{J_q(2\mathrm{i}|\xi|)}+1\right) \tag{25a}$$

$$\langle\xi;q|x_2^2|\xi;q\rangle=\frac{\hbar}{2m\omega}\left(2\mathrm{i}|\xi|\frac{J_{q-1}(2\mathrm{i}|\xi|)}{J_q(2\mathrm{i}|\xi|)}+1\right). \tag{25b}$$

In order to investigate in what sense these states are 'classical' it is useful to consider the classical limit ($\hbar\to 0$, $|\xi|\to\infty$: $\hbar\xi\to$ finite limit) of the above expectation values. Thus the relevant expectation values are given by

$$\langle\xi;q|x_1|\xi;q\rangle=0=\langle\xi;q|x_2|\xi;q\rangle \tag{26a}$$

$$\langle\xi;q|x_1^2|\xi;q\rangle=\frac{\hbar}{2m\omega}(2|\xi|+q+\tfrac{3}{2})+\mathrm{O}(\hbar/|\xi|) \tag{26b}$$

$$\langle\xi;q|x_2^2|\xi,q\rangle=\frac{\hbar}{2m\omega}(2|\xi|-q+\tfrac{3}{2})+\mathrm{O}(\hbar/|\xi|) \tag{26c}$$

$$\langle\xi;q|x_1x_2|\xi;q\rangle=\frac{\hbar}{m\omega}|\xi|\cos(2\theta+2\omega t). \tag{26d}$$

It may be observed that

$$\langle\xi;q|(x_1^2-x_2^2)|\xi;q\rangle=\frac{\hbar}{m\omega}q+\mathrm{O}(\hbar/|\xi|). \tag{27}$$

Equation (27), expressing the charge as the semiclassical approximation to the expectation value of $(x_1^2-x_2^2)$ provides the clue to the construction of the classical analogue. Since the energy of a classical oscillator is proportional to the square of the amplitude, the classical analogue of a state with definite charge is obtained by constraining the two oscillators described by the Hamiltonian

$$H=\frac{p_1^2}{2m}+\tfrac{1}{2}m\omega^2x_1^2+\frac{p_2^2}{2m}+\tfrac{1}{2}m\omega^2x_2^2 \tag{28}$$

to oscillate, keeping the difference of their action functions

$$\frac{E_1}{\omega}-\frac{E_2}{\omega}=\text{fixed}. \tag{29}$$

This quantity is taken to be proportional to the 'charge'. The role of this 'charge' in the motion of the classical oscillators is further clarified by performing a canonical transformation (Goldstein 1950), from (x_1, x_2, p_1, p_2) to (X_1, X_2, P_1, P_2), through the generating function

$$F(x_i, X_i)=\tfrac{1}{2}m\omega(x_1^2\cot X_1+x_2^2\cot X_2) \tag{30}$$

whence the transformed Hamiltonian becomes

$$K=H=\omega(P_1+P_2). \tag{31}$$

Thus X_i and P_i take the role of (phase) angle and action variables, and

$$P_1 - P_2 = \frac{E_1}{\omega} - \frac{E_2}{\omega} \tag{32}$$

is canonically conjugate to $\frac{1}{2}(X_1 - X_2)$ which is the relative phase between the two oscillators. Thus if one fixes the 'charge', namely the quantity $(P_1 - P_2)$, the various possible motions differ from each other in the relative phase φ of the two motions given by

$$x_1 = A\ \sin(\omega t + \theta + \varphi) \tag{33a}$$

$$x_2 = B\ \sin(\omega t + \theta - \varphi). \tag{33b}$$

If F_{cl} is a classical dynamical variable of this system with the 'charge' held fixed, the average, $\bar{F}_{\mathrm{cl}}$, of this quantity over all motions of the system in the state of given 'charge' (here $(P_1 - P_2)$) is obtained by taking an average (Messiah 1967) over the relative phase of the two oscillators (which is canonically conjugate to the 'charge'). Thus

$$\overline{x_1} = 0 = \overline{x_2} \tag{34a}$$

$$\overline{x_1^2} = \tfrac{1}{2}A^2 \tag{34b}$$

$$\overline{x_2^2} = \tfrac{1}{2}B^2 \tag{34c}$$

$$\overline{x_1 x_2} = AB\ \cos(2\theta + 2\omega t). \tag{34d}$$

Comparing these results with the classical limits of the quantal expectation values (equations (26)) we see that these are in agreement provided we identify $A^2 = B^2 = 2\hbar|\xi|/m\omega$. It may also be observed that $\langle X_1^2\rangle - \langle X_2^2\rangle \to 0$ in the classical limit and the 'charge' is thus a semiclassical quantity. It is in the sense discussed above that the modified coherent states for charged quanta discussed in this paper are 'classical'.

The coherent states for charged bosons have thus been constructed, their properties investigated and the corresponding classical model has been discussed.

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